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The Growth Estimate of Iterated Entire Functions in Terms of (p, q)-th Order

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1. Introduction, Definitions and Notation

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane **C**. It is well known [1], {[15], p-67, Th-1.46} that

$$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty \text{ and } \lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty.$$

After this Singh [11], Lahiri [7], Song and Yang [13], Singh and Baloria [12], Lahiri and Sharma [8] and Datta and Biswas [3], [4] proved different results on comparative growth property of composite entire functions. In a resent paper [2] Dutta study some comparative growth of iterated entire functions. In this chapter, we investigate the comparative growth of iterated entire functions in terms of its (p,q)-th order. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [5], [14] and [15].

The following definitions are well known.

Definition 1.1. The order ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r}.$$

Notation 1.2. [10] $\log^{[0]}x=x, \exp^{[0]}=x$ and for positive integer m, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

Definition 1.3. The p-th order ρ_f^p and lower p-th order λ_f^p of a meromorphic function $f(z)$ is defined as

$$\rho_f^p=\limsup_{r\to\infty}\frac{\log^{[p]}T(r,f)}{\log r}$$

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and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Clearly $\rho_f^p \le \rho_f^{p-1}$ and $\lambda_f^p \le \lambda_f^{p-1}$ for all *p* and when *p=1* then p-th order and lower p-th order coincide with classical order and lower order respectively.

Definition 1.4. The (p,q)-th order $\rho_f(p,q)$ and lower (p,q)-th order $\lambda_f(p,q)$ of a meromorphic function $f(z)$ is define as

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}.$$

If $f(z)$ is an entire function then

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$

where $p \ge q \ge 1$.

Clearly $\rho_f(p,1) = \rho_f^p$ *and* $\lambda(p,1) = \lambda_f^p$.

Definition 1.5. Let $f(z)$ be an entire function of finite p- th order ρ_f^p then we define σ_f^p as

$$\sigma_f^p = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{r^{\rho_f^p}}.$$

According to Lahiri and Banerjee [6] if $f(z)$ *and* $g(z)$ are entire functions then the iteration of f with respect to g is defined as follows:

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$$f_1(z) = f(z)$$
$$f_2(z) = f(g(z)) = f\left(g_1(z)\right)$$
$$f_3(z) = f\left(g\left(f(z)\right)\right) = f\left(g_2(z)\right) = f\left(g\left(f_1(z)\right)\right)$$
$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots$$
$$f_n(z) = f(g(f \ldots\ldots\ldots\ldots(f(z) \text{ or } g(z))\ldots\ldots\ldots)),$$

according as n is odd or even,

and so

$$g_1(z) = g(z)$$
$$g_2(z) = g(f(z)) = g\left(f_1(z)\right)$$
$$g_3(z) = g\left(f_2(z)\right) = g\left(f(g(z))\right)$$
$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots$$
$$g_n(z) = g\left(f_{n-1}(z)\right) = g\left(f\left(g_{n-2}(z)\right)\right).$$

Clearly all $f_n(z)$ *and* $g_n(z)$ *are entire functions.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have

$$T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. [1] If $f(z)$ and $g(z)$ are any two entire functions, for all sufficiently large values of r,

$$M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - \left|g(0)\right|, f\right) \le M(r, fog) \le M\left(M(r, g), f\right).$$

Lemma 2.3. [9] Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r, fog) \ge \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.4. Let $f(z)$ and $g(z)$ be two entire functions of non zero finite (p,q)-th order $\rho_f(p,q)$ and $\rho_g(p,q)$ respectively, then for any $\varepsilon > 0$ *and* $p \ge q \ge 1$,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \le \begin{cases} (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g) + O(1) & \text{when } n \text{ is even,} \\ (\rho_g(p,q) + \varepsilon) \log^{[q]} M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of r.

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Proof. First suppose that n is even. Then from second part of Lemma 2.2 and Definition of (p,q)-th order, it follows that for all sufficiently large values of r,

$$M(r, f_n) \leq M\left(M\left(r, g_{n-1}\right), f\right)$$

$$i.e., \ \log^{[p+1]} M(r, f_n) \leq \log^{[p+1]} M\left(M\left(r, g_{n-1}\right), f\right)$$

$$\leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g_{n-1}).$$

$$So, \ \log^{[p+2]} M(r, f_n) \leq \log^{[q+1]} M\left(r, g(f_{n-2})\right) + O(1)$$

$$\log^{[p+2-q]} M(r, f_n) \leq \log M\left(r, g(f_{n-2})\right) + O(1).$$

Taking repeated logarithms p times, we get

$$\log^{[2p+2-q]} M(r, f_n) \leq \log^{[p+1]} M(M(r, f_{n-2}), \mathrm{g}) + O(1)$$

$$\leq (\rho_g(p,q)+\varepsilon) \log^{[q]} M(r, f_{n-2}) + O(1),$$

$$i.e., \ \log^{[2p+3-q]} M(r, f_n) \leq \log^{[q+1]} M\left(r, f_{n-2}\right) + O(1)$$

$$\log^{[2p+3-2q]} M(r, f_n) \leq \log M\left(r, f_{n-2}\right) + O(1).$$

Again taking repeated logarithms p times, we get

$$\log^{[3(p+1)-2q]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, \mathrm{g}_{n-3}) + O(1).$$

Finally, after taking repeated logarithms *(n-4)(p+1)* times more, we have for all sufficiently large values of r,

$$\log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, \mathrm{g}) + O(1).$$

Similarly if n is odd then for all sufficiently large values of r,

$$\log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} M(r, f_n) \leq (\rho_g(p,q)+\varepsilon) \log^{[q]} M(r, f) + O(1).$$

This proves the lemma.

Lemma 2.5. Let $f(\mathrm{z})$ and $\mathrm{g}(\mathrm{z})$ be two entire functions of non zero finite lower (p,q)-th order $\lambda_f(p,q)$ *and* $\lambda_g(p,q)$ respectively, then for any

$$0 < \varepsilon < \min\{\lambda_f(p,q), \lambda_g(p,q)\} \ \ and \ \ p \geq q \geq 1,$$

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$$\log^{[(n-1)(p+1)-(n-2)q]} M(r,f_n) \geq \begin{cases} \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\dfrac{r}{2^{n-1}},g\right)+O(1) \text{ when } n \text{ is even} \\ \left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]} M\left(\dfrac{r}{2^{n-1}},f\right)+O(1) \text{ when } n \text{ is odd}, \end{cases}$$

for all sufficiently large values of r.

Proof. First suppose that n is even. Then from first part of Lemma 2.2 we have for all sufficiently large values of r and for any $0<\varepsilon<\min\{\lambda_f(p,q),\lambda_g(p,q)\}$,

$$M(r,f_n)=M\left(r,f(g_{n-1})\right)$$
$$\geq M\left(\frac{1}{8}M\left(\frac{r}{2},g_{n-1}\right)-\left|g_{n-1}(0)\right|,f\right)$$
$$\geq M\left(\frac{1}{16}M\left(\frac{r}{2},g_{n-1}\right),f\right).$$

$$\therefore \log^{[p+1]} M(r,f_n)\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{16}M\left(\frac{r}{2},g_{n-1}\right)\right],$$

using the Definition 1.4,

$$i.e.,\ \log^{[p+1]} M(r,f_n)\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2},g_{n-1}\right)+O(1)$$

$$i.e.,\ \log^{[p+2]} M(r,f_n)\geq\log^{[q+1]} M\left(\frac{r}{2},g(f_{n-2})\right)+O(1)$$

$$i.e.,\ \log^{[p+2-q]} M(r,f_n)\geq\log M\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right),g\right)+O(1).$$

Taking repeated logarithms p times, we get

$$\log^{[2p+2-q]} M(r,f_n)\geq\log^{[p+1]} M\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right),g\right)+O(1)$$
$$\geq\left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right)\right)+O(1)$$

$$\log^{[2p+3-2q]} M(r,f_n)\geq\log M\left(\frac{r}{2^2},f_{n-2}\right)+O(1).$$

Again taking repeated logarithms p times, we get

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$$\log^{[2p+2-2q]} M(r, f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{16}M\left(\frac{r}{2^3}, g_{n-3}\right)\right]+O(1)$$

$$\geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2^3}, g_{n-3}\right)+O(1).$$

Finally, after taking repeated logarithms *(n-4)(p+1)* times more, we have for all sufficiently large values of *r*,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{16}M\left(\frac{r}{2^{n-1}}, g\right)\right]+O(1)$$

$$i.e., \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right)+O(1).$$

Similarly if *n* is odd then for all sufficiently large values of *r*,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq \left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{2^{n-1}}, f\right)+O(1).$$

This proves the lemma.

Lemma 2.6. Let *f(z)* and *g(z)* be two non- constant entire functions, such that $0<\rho_f(p,q)<\infty$ *and* $0<\rho_g(p,q)<\infty$. Then for all sufficiently large *r* and $\varepsilon > 0$,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \leq \begin{cases} \left(\rho_f(p,q)+\varepsilon\right)\log^{[q]} M(r,g)+O(1) & \text{when } n \text{ is even,} \\ \left(\rho_g(p,q)+\varepsilon\right)\log^{[q]} M(r,f)+O(1) & \text{when } n \text{ is odd} \end{cases}$$

where $p \geq q \geq 1.$

The lemma follows from Lemma 2.1 and Lemma 2.4.

Lemma 2.7. Let $f(z)$ and $g(z)$ be two entire functions such that $0<\lambda_f(p,q)<\infty$ and $0<\lambda_g(p,q)<\infty.$ Then for any $\varepsilon(0<\varepsilon<\min\{\lambda_f(p,q), \lambda_g(p,q)\})$ and $p \geq q \geq 1,$

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq \begin{cases} \left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right)+O(1) & \text{when } n \text{ is even,} \\ \left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]} M\left(\frac{r}{4^{n-1}}, f\right)+O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of *r*.

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Proof. To prove this lemma we first suppose that n is even. Then from Lemma 2.1and Lemma 2.3 we get for any $\varepsilon(0<\varepsilon<\min\{\lambda_f(p,q),\ \lambda_g(p,q)\})$ and for all sufficiently large values of r,

$$T(r,f_n)=T\left(r,f(g_{n-1})\right)$$
$$\geq\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+\mathrm{O}(1),f\right).$$
$$\therefore\ \log^{[p]}T(r,f_n)\geq\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+\mathrm{O}(1),f\right)+\mathrm{O}(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+\mathrm{O}(1)\right]+\mathrm{O}(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{9}M\left(\frac{r}{4},g_{n-1}\right)\right]+\mathrm{O}(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q]}M\left(\frac{r}{4},g_{n-1}\right)+\mathrm{O}(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q-1]}T\left(\frac{r}{4},g_{n-1}\right)+\mathrm{O}(1)$$
$$\geq\left(\lambda_f(p,q)-\varepsilon\right)\log^{[q-1]}\left[\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+\mathrm{O}(1),\mathrm{g}\right)\right]+\mathrm{O}(1),$$

$$\text{i.e., }\log^{[p+1]}T(r,f_n)\geq\log^{[q+1]}M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+\mathrm{O}(1),\mathrm{g}\right)+\mathrm{O}(1)$$

$$\text{i.e., }\log^{[p+1-q]}T(r,f_n)\geq\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+\mathrm{O}(1),\mathrm{g}\right)+\mathrm{O}(1)$$

$$\text{i.e., }\log^{[2p+1-q]}T(r,f_n)\geq\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+\mathrm{O}(1),\mathrm{g}\right)+\mathrm{O}(1)$$
$$\geq\left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+\mathrm{O}(1)\right]+\mathrm{O}(1)$$
$$\geq\left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}\left[\frac{1}{9}M\left(\frac{r}{4^2},f_{n-2}\right)\right]+\mathrm{O}(1).$$

$$\text{i.e., }\log^{[2p+1-q]}T(r,f_n)\geq\left(\lambda_g(p,q)-\varepsilon\right)\log^{[q]}M\left(\frac{r}{4^2},f_{n-2}\right)+\mathrm{O}(1)$$

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$$\therefore\ \log^{[(n-1)p+(n-2)(1-q)]}T(r,f_n)\geq(\lambda_f(p,q)-\varepsilon)\log^{[q]}M\left(\frac{r}{4^{n-1}},g\right)+O(1)\ \textit{when } n \textit{ is even.}$$

Similarly,

$$\log^{[(n-1)p+(n-2)(1-q)]}T(r,f_n)\geq(\lambda_g(p,q)-\varepsilon)\log^{[q]}M\left(\frac{r}{4^{n-1}},f\right)+O(1)\ \textit{when } n \textit{ is odd.}$$

This proves the lemma.

3. Theorems

Theorem 3.1. Let *f* and *g* be two non-constant entire functions of non- zero finite (p,q)-th order and lower (p,q)-th order, also $0<\sigma_f^q,\sigma_g^q<\infty$. Then

$$(i)\quad \liminf_{r\to\infty}\frac{\log^{[(n-1)(p+1)-(n-2)q]}M(r,f_n)}{\log^{[p]}T(r,f(g))}\leq\frac{4^{\rho_g^q}\rho_f(p,q)}{\lambda_f(p,q)},$$

$$(ii)\quad \limsup_{r\to\infty}\frac{\log^{[(n-1)(p+1)-(n-2)q]}M(r,f_n)}{\log^{[p]}T(r,f(g))}\geq\frac{\lambda_f(p,q)}{\left(2^{n-1}\right)^{\rho_g^q}\rho_f(p,q)}$$

when *n* is even and

$$(iii)\quad \liminf_{r\to\infty}\frac{\log^{[(n-1)(p+1)-(n-2)q]}M(r,f_n)}{\log^{[p]}T(r,g(f))}\leq\frac{4^{\rho_f^q}\rho_g(p,q)}{\lambda_g(p,q)},$$

$$(iv)\quad \limsup_{r\to\infty}\frac{\log^{[(n-1)(p+1)-(n-2)q]}M(r,f_n)}{\log^{[p]}T(r,g(f))}\geq\frac{\lambda_g(p,q)}{\left(2^{n-1}\right)^{\rho_f^q}\rho_g(p,q)}$$

when *n* is odd.

Proof. First we suppose that *n* is even, then from Lemma 2.4 and the Definition 1.5 we have for all large *r* and $\varepsilon>0$,

$$\log^{[(n-1)(p+1)-(n-2)q]}M(r,f_n)\leq(\rho_f(p,q)+\varepsilon)\log^{[q]}M(r,g)+O(1)$$

Fro

$$\leq(\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right)r^{\rho_g^q}+O(1).\qquad(3.1)$$

m Lemma 2.3 we get

$$T(r,f(g))\geq\frac{1}{3}\log M\left(\frac{1}{16}M\left(\frac{r}{4},g\right),f\right).$$

Since $\lambda_f(p,q)$ is the lower (p,q)-th order of f so for given $\varepsilon(0<\varepsilon<\lambda_f(p,q))$ and for all large values of *r*,

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$$\log^{[p]} T(r, f(g)) \geq \log^{[p+1]} M\left(\frac{1}{16} M\left(\frac{r}{4}, g\right), f\right) + O(1)$$

$$\therefore \ \log^{[p]} T(r, f(g)) \geq \left(\lambda_f(p,q) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{4}, g\right) + O(1). \qquad (3.2)$$

Again for a sequence of values of *r* tending to infinity,

$$\log^{[q]} M\left(\frac{r}{4}, g\right) > \left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g^q}. \qquad (3.3)$$

Therefore from (3.2) and (3.3) we get for a sequence of values of *r* tending to infinity,

$$\log^{[p]} T(r, f(g)) \geq \left(\lambda_f(p,q) - \varepsilon\right)\left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g^q} + O(1) \qquad (3.4)$$

where $0 < \varepsilon < \min\{\lambda_f(p,q), \sigma_g^q\}$.

Now from (3.1) and (3.4) we have for a sequence of values of *r* tending to infinity,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{\left(\rho_f(p,q) + \varepsilon\right)\left(\sigma_g^q + \varepsilon\right) r^{\rho_g^q} + O(1)}{\left(\lambda_f(p,q) - \varepsilon\right)\left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g^q} + O(1)}$$

$$= \frac{\left(\rho_f(p,q) + \varepsilon\right)\left(\sigma_g^q + \varepsilon\right) 4^{\rho_g^q} + o(1)}{\left(\lambda_f(p,q) - \varepsilon\right)\left(\sigma_g^q - \varepsilon\right) + o(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{r \to \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{4^{\rho_g^q} \rho_f(p,q)}{\lambda_f(p,q)}.$$

Also when *n* is even then from Lemma 2.5 we get for all sufficiently large values of *r*

$$\log^{[(n-1)(p-1)-(n-2)q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1).$$

Now for a sequence of values of *r* tending to infinity, we have

$$\log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) > \left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q}.$$

Therefore for a sequence of values of *r* tending to infinity, we get

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon)\left(\sigma_g^q - \varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q} + O(1) \qquad (3.5)$$

where $0 < \varepsilon < \min\{\lambda_f(p,q), \sigma_g^q\}$.

Again by Lemma 2.1 we have for all large values of *r*, and $\varepsilon > 0$,

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$$T(r, f(g)) \le \log M(r, f(g))$$
$$\le \log M(M(r,g), f).$$
$$\therefore \ \log^{[p]} T(r, f(g)) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g)$$
$$\le (\rho_f(p,q)+\varepsilon)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q}. \qquad (3.6)$$

Therefore from (3.5) and (3.6) we have for a sequence of values of r tending to infinity,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \ge \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^q}+O(1)}{\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right) r^{\rho_g^q}}$$
$$= \frac{\left(\lambda_f(p,q)-\varepsilon\right)\left(\sigma_g^q-\varepsilon\right)+o(1)}{\left(2^{n-1}\right)^{\rho_g^q}\left(\rho_f(p,q)+\varepsilon\right)\left(\sigma_g^q+\varepsilon\right)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \ \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \ge \frac{\lambda_f(p,q)}{\left(2^{n-1}\right)^{\rho_g^q} \rho_f(p,q)}.$$

Similarly for odd n we get the second part of this theorem.
This proves the theorem.

Remark 3.2 If f is of regular growth i.e. $\rho_f(p, q) = \lambda_f(p, q)$ and n is even then

$$(i) \qquad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \le 4^{\rho_g^q},$$

$$(ii) \qquad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \ge \frac{1}{\left(2^{n-1}\right)^{\rho_g^q}}.$$

Also if g is of regular growth i.e. $\rho_g(p, q) = \lambda_g(p, q)$ and n is odd then

$$(iii) \qquad \liminf_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \le 4^{\rho_f^q},$$

$$(iv) \qquad \limsup_{r\to\infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \ge \frac{1}{\left(2^{n-1}\right)^{\rho_f^q}}.$$

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Remark 3.3. The conditions non zero lower (p,q)-th order and finite (p,q)-th order are necessary for Theorem 3.1, which are shown by the following examples.

Example 3.4. Let $f(z) = \exp^{[p-q+1]} z, g(z) = \exp^{[p-q]} z$ *and* $2q \geq p+1$. Then $\rho_f(p,q) = \lambda_f(p,q) = 1$ and $\rho_g(p,q) = \lambda_g(p,q) = 0$.

Here $f(g) = \exp^{[2p-2q+1]} z$ and

$$3T(2r, f(g)) \geq \log M(r, f(g)) = \exp^{[2p-2q]} r$$

$$i.e., \; T(r, f(g)) \geq \frac{1}{3} \exp^{[2p-2q]} \frac{r}{2}.$$

$$\therefore \; \log^{[p]} T(r, f(g)) \geq \exp^{[p-2q]} \frac{r}{2} + O(1).$$

Now

$$f_n = \begin{cases} \exp^{\left[n\,p - nq + \frac{n}{2}\right]} z & \text{when } n \text{ is even} \\ \exp^{\left[n\,p - nq + \frac{n+1}{2}\right]} z & \text{when } n \text{ is odd.} \end{cases}$$

So when n is even,

$$M(r, f_n) = \exp^{\left[n\,p - nq + \frac{n}{2}\right]} r$$

$$\therefore \; \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) = \log^{[(n-1)(p+1)-(n-2)q]} \exp^{\left[n\,p - nq + \frac{n}{2}\right]} r$$

$$= \exp^{\left[p - 2q - \frac{n}{2} + 1\right]} r.$$

Therefore

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{\exp^{[p-2q-\frac{n}{2}+1]} r}{\exp^{[p-2q]} r} + o(1)$$

$$= \frac{1}{\exp^{\left[\frac{n}{2}-1\right]} r} + o(1) \to 0 \; \textit{not gater then} \; 1 \text{ as } r \to \infty.$$

Similarly for odd n,

$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \leq \frac{\exp^{[p-2q-\frac{n-1}{2}+1]} r}{\exp^{[p-2q]} r} + o(1)$$

$$\to 0 \; \textit{not gater then} \; 1 \text{ as } r \to \infty.$$

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Example 3.5. Let $f(z) = \exp^{[p-q+1]}z, g(z) = \exp^{[p-q+2]}z$ and $2q \geq p+1$. Then $\rho_f(p, q) = \lambda_f(p, q) = 1$ and $\rho_g(p, q) = \lambda_g(p, q) = \infty$.

Here $g(f) = \exp^{[2p-2q+3]}z$ and

$$T(r, \mathrm{g}(f)) \leq \log \mathrm{M}(r, \mathrm{g}(f)) = \exp^{[2p-2q+2]}r$$

$$\therefore \log^{[p]} T(r, \mathrm{g}(f)) \leq \exp^{[p-2q+2]}r.$$

Now

$$f_n = \begin{cases} \exp^{\left[\mathrm{n}\, p - nq + \frac{3n}{2}\right]}z & when\, n\, is\, even \\ \exp^{\left[\mathrm{n}p - nq + \frac{3n-1}{2}\right]}z & when\, n\, is\, odd. \end{cases}$$

So when n is even,

$$M(r, f_n) = \exp^{\left[\mathrm{n}\, p - nq + \frac{3n}{2}\right]}r$$

i.e., $$\log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} M(r, f_n) = \log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} \exp^{\left[\mathrm{n}\, p - nq + \frac{3n}{2}\right]}r$$

$$= \exp^{\left[p - 2q + \frac{n}{2} + 1\right]}r.$$

Therefore

$$\frac{\log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} M(r, f_n)}{\log^{[p]} T\left(r, \mathrm{g}(f)\right)} \geq \frac{\exp^{\left[p - 2q + \frac{n}{2} + 1\right]}r}{\exp^{\left[p - 2q + 2\right]}r}$$

$$= \exp^{\left[\frac{n}{2} - 1\right]} r \to \infty\; not\; leas\; then 1 \text{ as } \mathrm{r} \to \infty.$$

When n is odd,

$$M(r, f_n) = \exp^{\left[\mathrm{n}\, p - nq + \frac{3n-1}{2}\right]}r$$

i.e., $$\log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} M(r, f_n) = \log^{[(\mathrm{n}-1)(\mathrm{p}+1)-(\mathrm{n}-2)\mathrm{q}]} \exp^{\left[\mathrm{n}\, p - nq + \frac{3n-1}{2}\right]}r$$

$$= \exp^{\left[p - 2q + \frac{n+1}{2}\right]}r.$$

Therefore

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$$\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} \geq \frac{\exp^{\left[p-2q+\frac{n+1}{2}\right]} r}{\exp^{[p-2q+2]} r}$$

$$= \exp^{\left[\frac{n+1}{2}-2\right]} r \to \infty \; not\; leas\; then 1 \text{ as } r \to \infty.$$

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