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Exploring the Concept of Bitranslations in Γ-Banach Spaces

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Abstract

Let $\xi(V) = \{p \mid p\colon \Gamma \to End(V)$ is an algebra homomorphism$\}$ for given any Γ-Banach space V over F. If $(x_1\mu x_2)_\gamma q = x_1\mu((x_2)_\gamma q), \forall\, x_1,\ x_2 \in V;\ \gamma,\mu \in \Gamma$ and $q \in \xi(V)$ is a right translation of V (in which case the argument is written on the left and $(\gamma)q$ denoted by ${}_\gamma q$), and $r \in \xi(V)$ is a left translation of V (in which case the argument is written on the right and $r(\gamma)$ is denoted by r_γ) if $r_\gamma(x_1\mu x_2) = \left(r_\gamma(x_1)\right)\mu x_2, \forall\, x_1,\ x_2 \in V;\ \gamma,\mu \in \Gamma$. $\xi_l(V)$ and $\xi_r(V)$ represent the set of all left and right translations of the Γ-Banach space V, respectively. If for any $x_1,\ x_2 \in V;\ \gamma,\mu \in \Gamma,\ x_1\gamma\left(r_\mu(x_2)\right) = (x_1)_\gamma q)\mu x_2$, then $p = (r,q) \in \xi_l(V) \times \xi_r(V)$ is a bitranslation of V. We investigate some properties of bitranslation of Γ- Banach Spaces in this paper.

Key Words: Γ- Banach Spaces, strong unities, bitranslation, Amicable, homothetism.

1. Introduction

First used by N. Nobusawa [1], the notation of a Γ-ring was somewhat reduced by W.E. Barnes [2], who also developed the concept of Γ-ring. When J. Luh [3], S. Kyuno [4], and W. E. Barnes [2] investigated the structure of Γ-ring, they found several generalizations that are comparable to related concepts in ring concept. R.C. Kalita, T.K. Dutta, and H.K. Nath [5] investigated the projective tensor product of Γ-Banach Algebras and Γ-derivations. Additionally, we study α-Characteristic Equations, α-Minimal Polynomial of Rectangular Matrices, K-derivation and symmetric bi-k-derivation on Gamma Banach Algebras [6], Operator Banach Algebras of Matrix Gamma Banach Agebras, Tensor product of modules, Gamma modules, and operator modules of Gamma modules. We examine a few characteristics of the bitranslation of Γ-Banach Spaces in this article.

2. Preliminaries

Definition 2.01 [1]Let V and Γ be two additive abelian groups and $V \times \Gamma \times V \to V$ and $\Gamma \times V \times \Gamma \to \Gamma$ be two maps such that

N1. $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$

N2. $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$ for all $x,\ y,\ z \in V$ and $\alpha,\beta \in \Gamma$.

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N3. If $x\alpha y = 0$ for all $x,\ y \in V$, then $\alpha = 0$.

Then V is called a Γ-ring in the sense of N. Nobusawa.

W. E. Barnes [2] define Γ-ring as follows:

Let V and Γ be two additive abelian groups. If there exists a mapping $V \times \Gamma \times V \to V$ (the image of (x, γ, y) where $x, y \in V$ *and* $\gamma \in \Gamma$, being denoted by $x\gamma y$), satisfying for all $x,\ y,\ z \in V$ and $\alpha, \beta \in \Gamma$:

B1. $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$
B2. $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$

Definition 2.02 Let V and Γ be additive abelian groups. Then V is called a Γ-Banach space over a field F if the following conditions hold:

1. V is a Γ-ring according to Barnes's definition.
2. V is a Banach space over F with a suitable norm.

Furthermore, according to Nobusawa's definition, if V is a Γ-ring, it is a Γ_N-Banach space over F.

Definition 2.03 A subset I of a Γ-Banach space V is said to be right (left) ideal of V if

(i) I is a subspace of V (in the vector space sense),
(ii) $x\alpha y \in I$ $(y\alpha x \in I)$ for all $x \in I,\ \alpha \in \Gamma,\ y \in V$ i.e. $I\Gamma V \subseteq I$ $(V\Gamma I \subseteq I)$.

A right Γ-ideal which is a left Γ-ideal as well as called a two sided Γ-ideal or simply a Γ-ideal. The notation $I \lhd V$ will mean I is a Γ-ideal of V.

Definition 2.04 Let $\xi(V) = \{p |\ p: \Gamma \to \mathrm{End}(V)$ is an algebra homomorphism$\}$ for given any Γ-Banach space V over F. If $(x_1\mu x_2)_\gamma q = x_1\mu\big((x_2)_\gamma q\big), \forall\, x_1,\ x_2 \in V;\ \gamma, \mu \in \Gamma$ and $q \in \xi(V)$ is a right translation of V (in which case the argument is written on the left and $(\gamma)q$ denoted by ${}_\gamma q$), and $r \in \xi(V)$ is a left translation of V (in which case the argument is written on the right and $r(\gamma)$ is denoted by r_γ) if $r_\gamma(x_1\mu x_2) = \big(r_\gamma(x_1)\big)\mu x_2, \forall\, x_1,\ x_2 \in V;\ \gamma, \mu \in \Gamma$. $\xi_l(V)$ and $\xi_r(V)$ represent the set of all left and right translations of the Γ-Banach space, respectively. If for any $x_1,\ x_2 \in V;\ \gamma, \mu \in \Gamma,\ x_1\gamma\left(r_\mu(x_2)\right) = (x_1)_\gamma q)\mu x_2$, then $p = (r, q) \in \xi_l(V) \times \xi_r(V)$ is a bitranslation of V. Then, $p = (r, q)$ is consider as a

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linked. $\xi_2(V)$ represents the set of all bitranslations of V. It shall be assumed that a bitranslation p is a double operator with $p(\gamma) = p_\gamma = r_\gamma$ and $(\gamma)p =_\gamma p =_\gamma q$. In particular, for any $x \in V$ define ${}^x p : \Gamma \to \mathrm{End}(V)$ and $q^x : \Gamma \to End(V)$ by ${}^x p(\gamma) = {}^x p_\gamma$ and $(\gamma) q^x = {}_\gamma q^x$, where ${}^x p_\gamma(y) = x\gamma y$ and $(y)_\gamma q^x = y\gamma x$ for all $x, y \in V$ and $\gamma \in \Gamma$. It is clear that ${}^x p$ is a left translation and q^x is a right translation of V. The pair $({}^x p, q^x)$ is a linked. Hence $({}^x p, q^x)$ is a bitranslation of V which we will denote by $[x] = ({}^x p, q^x)$. The inner bitranslation of V generated by x is the name given to the bitranslation $[x]$ that was constructed above. *I(V)* shall represent the set of all inner bitranslations of V. $E_l(V)$ represents the set of all endomorphisms of the form ${}^x p_\gamma$, while $E_r(V)$ represents the set of all endomorphisms of the form ${}_\gamma q^x$.

Definition 2.05 If for each $\gamma, \mu \in \Gamma$ and $x \in V$, $p_\gamma\big((x)_\mu q\big) = \big(p_\gamma(x)\big)_\mu q$ and $q_\gamma\big((x)_\mu p\big) = (q_\gamma(x))_\mu p$, then two bitranslations p and q of V are amicable. A collection of bitranslations of V for which every element is pairwise amiable is known as an amicable set.

Definition 2.06 A double homothetism p of a Γ- Banach space is a bitranslation of V, that is, amicable with itself.

Definition 2.07 A Γ-Banach space V is said to have a left (right) strong unity if there exists some $d \in V$, $\delta \in \Gamma$ such that $d\delta x = x$ $(x\delta d = x)$, for all $x \in V$.

Definition 2.06: A Γ- Banach space V has a left (right) double homothetism strong unity if there exists double homothetism p of V and $\gamma \in \Gamma$ such that $p_\gamma(x) = x$ $((x)_\gamma p = x)$ for all $x \in V$.

3. Main Results

Theorem 3.01 A Γ- Banach space has strong unities on the left and right if and only if it has only inner double homothetisms and left and right double homothetisms strong unities.

Proof: Assume that V has left and right strongly unities, meaning that for any m in V, $x\alpha m = m$ and $m\beta y = m$, where $x \in V, \alpha \in \Gamma$ and $y \in V, \beta \in \Gamma$.Then $[x] \in I(V)$ for every $x \in V$ and $[x]_\alpha(m) = x\alpha m = m$ for all $m \in V$. A left double homothetism strong unity is thus formed by the double homothetism $[x]$ of V and $\alpha \in \Gamma$. In the same way, $\beta \in \Gamma$ and V's

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double homothetism $[y]$ generate a right double homothetism strong unity for V. Consider any double homothetism of V to be p. For $m \in V, \mu \in \Gamma$ we have $p_\mu(m) = x\alpha\left(p_\mu(m)\right) = [(x)_\alpha p]\mu m = [(x)_\alpha p]_\mu(m)$. Also $(m)_\mu p = \left[(m)_\mu p\right]\beta y = m_\mu[p_\beta(y)]$.

Thus $(x)_\alpha p = [(x)_\alpha p]\beta y = x\alpha[p_\beta(y)] = p_\beta(y)$. Because of $(x)_\alpha p = p_\beta(y)$, V has an inner double homothetism p, that is, V only possesses inner double homothetisms.

Let V, on the other hand, has left and right double homothetism strong unities, and only inner double homothetisms are present. Suppose that $\gamma \in \Gamma$ and that p is the double homothetism, such that $p_\gamma(m) = m$. Due to the fact that V only has inner double homothetism, there exists $e \in V$ such that $p = [e]$. Likewise, with q representing the double homothetism and $\mu \in \Gamma$, so that $(m)_\mu q = m$. Since V has only inner double homothetisms, there exists $a \in V$ such that $q = [a]$.

If $m \in V$, then $e\gamma m = [e]_\gamma(m) = p_\gamma(m) = m$ and $m\gamma a = (m)_\mu[a] = (m)_\mu q = m$. Thus the Γ-Banach space V has left and right strong unities.

Theorem 3.02 If V is a Γ-Banach space that has a left or a right strong unity, then V is isomorphic to the Γ-Banach space *I(V)* of all inner double homothetism of V.

Proof: Define the mapping $f: V \to I(V)$ by $f(m) = [m]$, $m \in V$. Straight forward calculations will show that f is a onto Γ-Banach space homomorphism. Let $e \in V$ and $\gamma \in \Gamma$ form a right strong unity of V. Let $m_1, m_2 \in V$ such that

$$f(m_1) = f(m_2) \Rightarrow [m_1] = [m_2] \Rightarrow [m_1]_\gamma(e) = [m_2]_\gamma(e) \Rightarrow m_1\gamma e = m_2\gamma e \Rightarrow m_1 = m_2$$

Hence f is an isomorphism from V onto *I(V)*. The result follows similarly if V has a left strong unity.

Theorem 3.03 Let V be any Γ-Banach space. V has a left (right) double homothetism strong unity if it can be embedded as an ideal in a Γ-Banach space with left (right) strong unity.

Proof: Assume that V may be embedded as an ideal in a left-strong unity Γ-Banach space U. Then there exist $e \in U$ and $\gamma \in \Gamma$ such that $e\gamma n = n$ for all $n \in U$. Since $e \in U$, so $[e] = ({}^e p, q^e)$ where ${}^e p$ and q^e are left and right translation of U respectively with ${}^e p_\gamma(n) = e\gamma n$

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and $(n)_\gamma q^e = n\gamma e$ for all $n \in U$. Let $m \in V$. Then ${}^e p_\gamma(m) = e\gamma m \in V$ and $(m)_\gamma q^e = m\gamma e \in V$, because $V < V'$. Hence the restrictions of both ${}^e p$ and q^e to V say ${}^e p|V$ and $q^e|V$ are left and right translations respectively of V. Let $p = \left({}^e p|V, q^e|V\right)$ be a double homothetism of V. Then $p_\gamma(m) = [e]_\gamma(m)$ for all $m \in V$ ·This demonstrates that the double homothetism p of V, $\gamma \in \Gamma$ forms a left double homothetism strong unity of V. Similar reasoning demonstrates that if U has a right strong unity, then V must also have a right double homothetism strong unity.

Lemma 3.04 For a field *F*, let *V* be a Γ-Banach space over a field *F*. Then, both the sets $\xi_l(V)$ and $\xi_r(V)$ of all left and right translations of *V* respectively are Γ-Banach space over field *F*.

Theorem 3.05 A Γ-Banach space *V* can be embedded as an ideal in a Γ-Banach space with a left (right) strong unity if the Γ-Banach space *V* has left (right) double homothetism strong unity.

Proof: Assume that *V* is a Γ-Banach space that has a left double homothetism strong unity. Let E_l be the direct sum of $\xi_l(V)$ (the set of all left translations of $V_{[1]}$) and *V*, that is, $E_l = \xi_l(V) \oplus V$· Define a map $E_l \times \Gamma \times E_l \to E_l$ by $(p^1, m_1)\gamma(p^2, m_2) = (p^1\gamma p^2, m_1\gamma m_2)$ for all $(p^1, m_1)\gamma(p^2, m_2) \in E_l$. Since $\xi_l(V)$ and V are both Γ-rings, So $E_l = \xi_l(V) \oplus V$ is also Γ-ring. We can show that E_l is a vector space over *F*. Next define a norm on E_l by $\|(p, m)\| = \|p\| + \|m\|$.

It can be easily show that $(E_l, \|.\|)$ is a normed linear space. If $\{(p^n, x_n)\}$ is a Cauchy sequence in E_l, then for given $\varepsilon > 0$ there exists a positive integer n_0 such that for $m, n \geq$ *n0, (pn, xn)−(pm, xm)<ε*

$$\Rightarrow \left\| \left(p^n - p^m, x_n - x_m\right) \right\| < \varepsilon$$

$$\Rightarrow \left\|p^n - p^m\right\| + \left\|x_n - x_m\right\| < \varepsilon$$

$$\Rightarrow \left\|p^n - p^m\right\| < \frac{\varepsilon}{2} \text{ and } \left\|x_n - x_m\right\| < \frac{\varepsilon}{2}$$

$\Rightarrow \{p^n\}$ and $\{x_n\}$ are Cauchy sequence in $\xi_l(V)$ and *V* respectively.

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Since $\xi_l(V)$ and V are Γ-Banach algebras over F, therefore there exist $p \in \xi_l(V)$ and $x \in V$ such that $p^n \to p$ and $x_n \to x$. Then we can prove easily that $(p^n, x_n) \to (p, x)$. Hence E_l is a Banach space. Hence E_l is a Γ-Banach space.

The subset $U = \{(0, m) | m \in V\}$ of E_l is an ideal of E_l and is isomorphic (as a Γ-Banach space) to V. Let the double homothetism p of V and $\gamma \in \Gamma$ form a left double homothetism strong unity of V. By theorem 6.1, there exists $e \in V$ such that $p = [e]$ and $e\gamma m = m\gamma e = m$ for all $m \in V$. Then for any $\in \xi_l(V), m \in V, (p\gamma q)_\gamma(m) = (p_\gamma \circ q_\gamma)(m) = p_\gamma\left(q_\gamma(m)\right) = q_\gamma(m)$..Thus $(p\gamma q) = q$ for all $q \in \xi_l(V)$. Also $(p, e) \in E_l$.

If (q, m) is an element of E_l, then $(p, e)\gamma(q, m) = (p\gamma q, e\gamma m) = (q, m)$. Thus $(p, e) \in E_l$ and $\gamma \in \Gamma$ form a left strong unity of E_l. Hence V can be embedded as an ideal in a Γ-Banach space with left strong unity. Similarly, if V has a right double homothetism strong unity, then V is isomorphic to an ideal of $E_r = V \oplus \xi_r(V)$. Which is a Γ-Banach space with right strong unity, where $\xi_r(V)$ is the set of all right translations of V.

Conclusion: This study is grounded in fundamental concepts of rings, linear spaces, Banach spaces, Banach algebras, Γ-rings, and Γ-Banach space. It Explores the Concept of Bitranslations in Γ-Banach Spaces. The investigation proved to be insightful and contributed to a deeper understanding of the subject. It is hoped that this work will motivate further research in this area. The authors also believe that the findings may be expanded in the future through alternative methods.

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